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On Some Nikolskiĭ- and Oswald-type Inequalities

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Inequalities of Nikolskiĭ (Trudy Mat. Inst. Steklova 38 (1951), 2.3, p. 255) in $L^p_{2\pi}$, $p \geq 1$, and of Oswald (Izv. Vyssh. Uchebn. Zaved. Mat. 7 (1976), (3.4), p. 71; Theorem 1, p. 69), $0 < p < 1$, are extended to the case of Orlicz spaces $L^\varphi_{2\pi}$.

1

Let $\varphi\colon \langle 0, 2\pi) \times R_+ \to R_+$ be a function of class Φ (see [3, 7.1]), generating the generalized Orlicz space $L^\varphi_{2\pi}$ and convex with respect to the second variable and let us extend φ to $R \times R_+$ 2π-periodically. The function φ is said to satisfy the condition (A), if there is a set $A^0 \subset R$, $\operatorname{mes} A^0 = 0$, and numbers $\bar{c} > 0$, $M > 0$ and a measurable function $F_1\colon (R \backslash A^0) \times (R \backslash A^0) \to R_+$ such that $\varphi(x, u) \leq \varphi(t, \bar{c}u) + F_1(t, x)$ for all $u \geq 0$, $t, x \in R \backslash A^0$, and $\iint_Q F_1(t, x)\, dt\, dx \leq M \operatorname{mes} Q$ for every square Q in $\langle 0, 2\pi \rangle \times \langle 0, 2\pi \rangle$. The function φ is said to satisfy the condition (B_η) with an $\eta > 0$, if there exist a set $A \subset \langle 0, 2\pi \rangle$, $\operatorname{mes} A = 0$, a constant $c > 0$, and a nonnegative, 2π-periodic, measurable function $F(\cdot, h)$ on R for $|h| \leq \eta$, satisfying the inequality $S_\eta = \sup_{|h| \leq \eta} \int_0^{2\pi} F(t, h)\, dt < \infty$ such that $\varphi(t - h, u) \leq \varphi(t, cu) + F(t, h)$ for $u \geq 0$, $t \in \langle 0, 2\pi \rangle \backslash A$ (see [1]). It is easily seen that if φ satisfies B_π, then it satisfies also (B_η) for any $\eta > \pi$ with the same set A, constant c and $S_\eta = S_\pi$. Obviously, if $\varphi(t, u)$ is independent of the parameter t, it satisfies both (A) and (B_η) for all $\eta > 0$.

2

The following notation will be adopted. Taking a positive integer N fixed and $x_j = 2\pi j N^{-1}$ for $j = 0, 1, \ldots, N-1$, we write for any N-dimensional real vector $\bar{v} = v_0, \ldots, v_{N-1}$,

$$\rho_\varphi^{(N)}(\bar{v}) = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \varphi(t, |v_j|)\, dt.$$

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This is a convex modular in R^N, and the Luxemburg norm $\| \|_{\rho_\varphi^{(N)}}$ generated by this modular ([3, 1.5]) will be further denoted by $\| \|_\varphi^{(N)}$. Let T_n be a trigonometric polynomial of degree $\leqslant n$, and let $\bar{v} \in R^N$, $T_n \bar{v} = (T_n(v_0), ..., T_n(v_{N-1}))$. Thus, $\rho_\varphi^{(N)}(T_n \bar{v})$ defines a convex pseudomodular in the space H_n of all trigonometric polynomials of degree $\leqslant n$, generating a pseudonorm $\|T_n \bar{v}\|_\varphi^{(N)}$, $\bar{v}$ being fixed. We define in H_n also another convex modular

$$\rho_\varphi^N(T_n) = \sup_x \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \varphi(t, |T_n(x + x_j)|)\, dt,$$

defining in H_n the Luxemburg norm $\| \|_\varphi^N = \| \|_{\rho_\varphi^N}$.

3

The purpose of the first part of this note is to estimate the pseudonorm $\|T_n \bar{x}\|_\varphi^{(N)}$ with $\bar{x} = x_0, ..., x_{N-1}$ and the norm $\|T_n\|_\varphi^N$, by means of the Luxemburg norm $\|T_n\|_\varphi$ of T_n in the generalized Orlicz space $L_{2\pi}^\varphi$, generated by the modular $\rho(f) = \int_0^{2\pi} \varphi(t, |f(t)|)\, dt$.

LEMMA 1. *Let φ be a convex function of the class Φ, satisfying* (A) *and* (B_π), *and let $T_n \in H_n$. Then there holds*

$$\|T_n \bar{x}\|_\varphi^{(N)} \leqslant (1 + 2\pi n N^{-1} C_1) \|T_n\|_\varphi, \tag{1}$$

where $C_1 = 2\bar{c}c \max(1, S_\pi + 2\pi M)$. *In case of $\varphi(t, u)$ independent of the parameter t,*

$$\|T_n \bar{x}\|_\varphi^{(N)} \leqslant (1 + 2\pi n N^{-1}) \|T_n\|_\varphi. \tag{2}$$

Proof. Let $\eta_j \in \langle x_j, x_{j+1} \rangle$ be chosen so that $|T_n(\eta_j)| = \min_{x_j \leqslant t \leqslant x_{j+1}} |T_n(t)|$, $\bar{\eta} = \eta_0, ..., \eta_{N-1}$. Then $\rho_\varphi^N(u^{-1} T_n \bar{\eta}) \leqslant \rho_\varphi(u^{-1} T_n)$ for every $u > 0$, whence $\|T_n \bar{\eta}\|_\varphi^{(N)} \leqslant \|T_n\|_\varphi$. Hence

$$\|T_n \bar{x}\|_\varphi^{(N)} \leqslant \|T_n \bar{x} - T_n \bar{\eta}\|_\varphi^{(N)} + \|T_n \bar{\eta}\|_\varphi^{(N)} \leqslant \|T_n \bar{x} - T_n \bar{\eta}\|_\varphi^{(N)} + \|T_n\|_\varphi. \tag{3}$$

Let $u > 0$ and $d \geqslant 1$ be arbitrary. Then

$$\rho_\varphi^{(N)} \left(\frac{T_n \bar{x} - T_n \bar{\eta}}{du} \right) \leqslant \frac{1}{d} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \varphi \left\{ t, \frac{1}{u} \int_{x_j}^{x_{j+1}} |T_n'(s)|\, ds \right\} dt.$$

Now, by Jensen's inequality with a fixed t and by condition (A), we obtain

$$\varphi \left\{ t, \frac{1}{u} \int_{x_j}^{x_{j+1}} |T_n'(s)|\, ds \right\} \leqslant \frac{N}{2\pi} \int_{x_j}^{x_{j+1}} \varphi \left\{ s, \frac{2\pi \bar{c}}{Nu} |T_n'(s)| \right\} ds$$
$$+ \frac{N}{2\pi} \int_{x_j}^{x_{j+1}} F(s, t)\, ds.$$

Now, under the assumption (B_π) there holds the following Bernstein inequality: $\rho_\varphi(n^{-1}T_n') \leqslant \rho_\varphi(cT_n) + S_\pi$ (see [1, Proposition 2]). Hence we obtain easily

$$\rho_\varphi^{(N)}\left(\frac{T_n\bar{x} - T_n\bar{\eta}}{du}\right) \leqslant \rho_\varphi\left(\frac{2\pi\bar{c}c}{Nu} T_n\right) + \frac{1}{d} C,$$

where $C = S_\pi + 2\pi M$. Now let $d = \max(1, C)$. Then

$$\rho_\varphi^{(N)}\left(\frac{T_n\bar{x} - T_n\bar{\eta}}{2du}\right) \leqslant \frac{1}{2}\rho_\varphi\left(\frac{2\pi\bar{c}c}{Nu} T_n\right) + \frac{1}{2}. \tag{4}$$

From this inequality follows that if $u > 2\pi\bar{c}cnN^{-1}$, then the left-hand side of the inequality (4) is $\leqslant 1$. Hence

$$\|T_n\bar{x} - T_n\bar{\eta}\|_\varphi^{(N)} \leqslant 4\pi\bar{c}cdnN^{-1} \|T_n\|_\varphi.$$

From this and from (3) follows (1). If φ does not depend on t, we have $M = S_\pi = 0$, $c = \bar{c} = d = 1$, and we get (2).

THEOREM 1. *Let φ be a convex function of the class Φ, satisfying* (A) *and* (B_π), *and let $T_n \in H_n$. Then*

$$\|T_n\|_\varphi^N \leqslant (1 + 2\pi nN^{-1}C_1)\, C_2 \|T_n\|_\varphi, \tag{5}$$

where C_1 is the same as in Lemma 1 *and $C_2 = 2c\max(1, S_\pi)$ for $S_\pi \geqslant \frac{1}{2}$, $C_2 = c(1 - S_\pi)^{-1}$ for $0 \leqslant S \leqslant \frac{1}{2}$. In case of $\varphi(t, u)$ independent of t we have*

$$\|T_n\|_\varphi^N \leqslant (1 + 2\pi nN^{-1}) \|T_n\|_\varphi.$$

Proof. We apply Lemma 1 to $S_n(\cdot) = T_n(x + \cdot)$ with fixed x, obtaining $\|S_n\bar{x}\|_\varphi^{(N)} \leqslant (1 + 2\pi nN^{-1}C_1) \|S_n\|_\varphi$. However, due to the assumption (B_π) we have $\|S_n\|_\varphi \leqslant C_2\|T_n\|_\varphi$ (see [2, Theorem 1]), whence

$$\|S_n\bar{x}\|_\varphi^{(N)} \leqslant (1 + 2\pi nN^{-1}C_1)\, C_2 \|T_n\|_\varphi,$$

$$\rho_\varphi^{(N)}\left(\frac{\delta S_n\bar{x}}{(1 + 2\pi nN^{-1}C_1)\, C_2 \|T_n\|_\varphi}\right) \leqslant \delta < 1 \qquad \text{for} \quad 0 < \delta < 1.$$

Passing to the limit as $\delta \to 1$ and then taking supremum over x, we get

$$\rho_\varphi^{(N)}\left(\frac{T_n}{(1 + 2\pi nN^{-1}C_1)\, C_2 \|T_n\|_\varphi}\right) \leqslant 1,$$

which yields (5). If φ does not depend on t, we see easily that we may take $C_1 = C_2 = 1$.

Let us remark that if $\varphi(t, u)$ is independent of t, then obviously $\rho_\varphi(T_n) \leqslant \rho_\varphi^N(T_n)$, whence $\|T_n\|_\varphi \leqslant \|T_n\|_\varphi^N$. This gives

COROLLARY. *If φ is a convex φ-function (see [3, 1.9]) independent of the parameter and if $T_n \in H_n$, then*

$$\|T_n\|_\varphi \leqslant \|T_n\|_\varphi^N \leqslant (1 + 2\pi n N^{-1}) \|T_n\|_\varphi.$$

If $\varphi(u) = |u|^p$, $p \geqslant 1$, the corollary gives exactly the Nikolskiĭ inequalities in case of one variable.

4

We shall now assume φ to be a concave φ-function without parameter, strongly s-convex with an $s \in (0, 1\rangle$, i.e., $\tilde{\varphi}(u) = \varphi(u^{1/s})$ is a convex function. Then for $T_n \in H_n$ we have

$$\rho_\varphi^N(T_n) = \frac{2\pi}{N} \sup_x \sum_{j=0}^{N-1} \varphi(|T_n(x + x_j)|);$$

by $\|T_n\|_\varphi^{N,s}$ we denote the respective s-homogeneous norm in H_n (see [3, 1.5]). Besides $\| \ \|_\varphi^{N,s}$ we shall consider in H_n also the s-homogeneous norm $\| \ \|_\varphi^s$ induced in H_n by $L_{2\pi}^\varphi$ generated by φ. The following is easily calculated:

LEMMA 2. *If $\tilde{\psi}$ is a convex φ-function and*

$$K_{1,n}(t) = \left(\frac{\sin(1/2)\, nt}{n \sin(1/2) t}\right)^2 \qquad \text{for} \quad 0 < t < 2\pi,$$

then for every $C > 0$,

$$\int_0^{2\pi} \tilde{\psi}(CK_{1,n}(t))\, dt \leqslant \tfrac{1}{2}\pi^3 \psi(4C\pi^{-2})\, n^{-1}.$$

THEOREM 2. *Let φ be a concave, strongly s-convex φ-function without parameter, $0 < s \leqslant 1$, satisfying the condition (Δ_2): $\psi(u) = \sup_{v>0} \varphi(uv)/\varphi(v) < \infty$ for all $u > 0$. Let $\tilde{\psi}(u) = \psi(u^{(2r-1)/2})$, $u \geqslant 0$, with an integer $r \geqslant (s+2)/2s$. Then for every $T_n \in H_n$,*

$$\rho_\varphi(T_n) \leqslant \rho_\varphi^N(T_n) \leqslant 2^r \tilde{\psi}(1 + 2\pi n N^{-1})\, \rho_\varphi(T_n),$$

$$\|T_n\|_\varphi^s \leqslant \|T_n\|_\varphi^{N,s} \leqslant 2^r \tilde{\psi}(1 + 2\pi n N^{-1})\, \|T_n\|_\varphi^s.$$

Proof. The left-hand side inequalities follow as in the remark to Theorem 1. To prove the right-hand ones, we denote $t_k^{(n)}=(2k+1)\pi/2n$ for $k=0, 1,..., 2n-1$. Then, applying Lemma 2 from [5, 1.7, p. 68], subadditivity of φ and the definition of ψ, we obtain

$$\frac{2\pi}{N}\sum_{j=0}^{N-1}\varphi(|T_n(x+x_j)|)$$
$$\leqslant\frac{2\pi}{N}\sum_{j=0}^{N-1}\varphi\left\{\sum_{k=0}^{2^rn-1}|T_n(t_n^{(2^{r-1}n)})|\,|K_{1,n}(x+x_j+t_k^{(2^{r-1}n)})|^{(2r-1)/2}\right\}$$
$$\leqslant\sum_{k=0}^{2^rn-1}\varphi(|T_n(t_n^{(2^{r-1}n)})|)\frac{2\pi}{N}\sum_{j=0}^{N-1}\psi(|K_{1,n}(x+x_j-t_k^{(2^{r-1}n)})|^{(2r-1)/2}).$$

Since $r\geqslant(s+2)/2s$, so $p=2/(2r-1)\leqslant 1$. Since φ is strongly s-convex, so is ψ; hence ψ is also strongly p-convex, whence $\tilde{\psi}$ is convex. Moreover,

$$\frac{2\pi}{N}\sum_{j=0}^{N-1}\psi(|K_{1,n}(x+x_j-t_k^{(2^{r-1}n)})|^{(2r-1)/2})\leqslant\rho_{\tilde{\psi}}^N(K_{1,n}).$$

Hence

$$\frac{2\pi}{N}\sum_{j=0}^{N-1}\varphi(|T_n(x+x_j)|)\leqslant\rho_{\tilde{\psi}}^N(K_{1,n})\sum_{k=0}^{2^rn-1}\varphi(|T_n(t_k^{(2^{r-1}n)})|)$$

for every x. Now, let $\eta_j\in\langle x_j, x_{j+1}\rangle$ be as in the proof of Lemma 1 and let $\rho_\varphi^{(N)}(\bar{v})$ be as in **2**, with $\tilde{\psi}$ in place of φ. Then $\rho_{\psi}^{(N)}(T_n\bar{\eta})\leqslant\rho_\psi(T_n)$. Hence

$$\rho_{\tilde{\psi}}^{(N)}(T_n\bar{x})\leqslant\tfrac{1}{2}\rho_{\tilde{\psi}}^{(N)}(2T_n\bar{x}-2T_n\bar{\eta})+\tfrac{1}{2}\rho_{\tilde{\psi}}(2T_n).$$

Calculating as in the proof of Lemma 1 with $d=1$ and $u=\frac{1}{2}$ and applying Bernstein inequality, we obtain

$$\rho_{\tilde{\psi}}^{(N)}(2T_n\bar{x}-2T_n\bar{\eta})\leqslant\rho_{\tilde{\psi}}(4\pi nN^{-1}T_n).$$

Hence

$$\rho_{\tilde{\psi}}^{(N)}(T_n\bar{x})\leqslant\tfrac{1}{2}\rho_{\tilde{\psi}}(4\pi nN^{-1}T_n)+\tfrac{1}{2}\rho_{\tilde{\psi}}(2T_n).$$

Applying this inequality to $S_n(\cdot)=T_n(x+\cdot)$ with a fixed x, we obtain

$$\frac{2\pi}{N}\sum_{j=0}^{N-1}\tilde{\psi}(|T_n(x+x_j)|)\leqslant\tfrac{1}{2}\rho_{\tilde{\psi}}(4\pi nN^{-1}T_n)+\tfrac{1}{2}\rho_{\tilde{\psi}}(2T_n).$$

Taking supremum over all x, we get

$$\rho_{\tilde{\psi}}^N(T_n)\leqslant\tfrac{1}{2}\rho_{\tilde{\psi}}(4\pi nN^{-1}T_n)+\tfrac{1}{2}\rho_{\tilde{\psi}}(2T_n)$$

for every $T_n \in H_n$. Now, we apply this inequality to $K_{1,n} \in H_n$ in place of T_n. By Lemma 2 and superadditivity of $\tilde{\psi}$, we thus obtain

$$\rho_{\tilde{\psi}}^N(K_{1,n}) \leqslant \frac{\pi^3}{4n}\left(\tilde{\psi}\left(\frac{16n}{N\pi}\right)+\tilde{\psi}\left(\frac{8}{\pi^2}\right)\right) \leqslant \frac{2\pi}{n}\tilde{\psi}\left(1+\frac{2\pi n}{N}\right).$$

Consequently,

$$\frac{2\pi}{N}\sum_{j=0}^{N-1}\varphi(|T_n(x+x_j)|) \leqslant \frac{2\pi}{n}\tilde{\psi}\left(1+\frac{2\pi n}{N}\right)\sum_{k=0}^{2^r n-1}\varphi(|T_n(t_k^{(2^{r-1}n)})|).$$

Taking supremum over all x, we obtain

$$\rho_\varphi^N(T_n) \leqslant 2^r\tilde{\psi}\left(1+\frac{2\pi n}{N}\right)\frac{2\pi}{2^r n}\sum_{k=0}^{2^r n-1}\varphi(|T_n(t_k^{(2^{r-1}n)})|).$$

Writing $S_n(\cdot) = T_n(x+\cdot)$ for an arbitrary x, we have obviously $\rho_\varphi^N(S_n) = \rho_\varphi^N(T_n)$. Hence, applying the above inequality to S_n in place of T_n, we get

$$\rho_\varphi^N(T_n) = \rho_\varphi^N(S_n) \leqslant 2^r\tilde{\psi}\left(1+\frac{2\pi n}{N}\right)\frac{2\pi}{2^r n}\sum_{k=0}^{2^r n-1}\varphi(|T_n(x+t_k^{(2^{r-1}n)})|).$$

Integrating both sides over $\langle 0, 2\pi\rangle$, we obtain

$$2\pi\rho_\varphi^N(T_n) \leqslant 2^{r+1}\tilde{\psi}(1+2\pi n N^{-1})\,\rho_\varphi(T_n),$$

which is the first of the required inequalities. The second inequality follows easily from the first one.

Let us remark that taking $\varphi(u) = |u|^p$, $0 < p < 1$, Theorem 2 yields the inequalities of Oswald [5, 3.4, p. 71].

5

THEOREM 3. *Let φ be a concave, s-convex function (see [3, 1.9.I]) depending on the parameter, satisfying* (B_π) *and the condition* (Δ_2): $\psi(t, u) = \sup_{v>0}\varphi(t, uv)/\varphi(t, v) < \infty$ *for all* $u \geqslant 0$ *and* $t \in \langle 0, 2\pi\rangle$. *Then there exists a* $C > 0$ *such that for every* $T_n \in H_n$ *there holds*

$$\|T_n^{\nu}\|_\varphi^s \leqslant C^s n^{s\nu}\|T_n\|_\varphi^s \qquad \text{for} \quad \nu = 0, 1, 2, \ldots.$$

Proof. Obviously, it is sufficient to perform the proof for $\nu = 1$. Since φ is s-convex, so are ψ and $\tilde{\psi}(u) = \sup_{0\leqslant t\leqslant 2\pi}\psi(t, u)$, $\tilde{\psi}(1) = 1$. Choosing a fixed positive integer r such that $2rs > 1$, we thus obtain

$$\sum_{k=0}^{\infty}\tilde{\psi}\left(\frac{1}{(2k+1)^{2r}}\right) \leqslant \sum_{k=1}^{\infty}\frac{1}{(2k+1)^{2rs}}\tilde{\psi}(1) < \infty. \tag{6}$$

Taking t_k^n as in the proof of Theorem 2 and applying the inequality

$$|T_n'(x)| \leqslant \sum_{k=0}^{2^r n-1} \frac{2^r n}{4n^{2r}} |T_n(x+t_k^{(2^{r-1}n)})| (\sin \tfrac{1}{2} t_k^{(2^{r-1}n)})^{-2r}$$

(see [5, p. 69]), subadditivity of φ and inequality (6) give for every $\lambda > 0$,

$$\rho_\varphi(\lambda T_n') \leqslant 2 \sum_{k=0}^{\infty} \psi\left(\frac{1}{(2k+1)^{2rs}}\right) \rho_\varphi(\lambda 2^{2r^2+r-2} n T_n(\cdot + t_k^{(2^{r-1}n)})). \tag{7}$$

Now, by [2, Theorem 1], we have

$$\|\lambda 2^{2r^2+r-2} n T_n(\cdot + t_k^{(2^{r-1}n)})\|_\varphi^s \leqslant C_2 \lambda^s 2^{(2r^2+r-2)s} n^s \|T_n\|_\varphi^s,$$

where C_2 is as in Theorem 1. Choosing

$$\lambda = \{2^{2r^2+r-2} n C_2^{1/s} (\|T_n\|_\varphi^s)^{1/s}\}^{-1}, \tag{8}$$

the left-hand side of the last inequality becomes $\leqslant 1$ and so, by (7), we obtain

$$\rho_\varphi(\lambda T_n') \leqslant 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2rs}}.$$

If C_3 is the maximum of 1 and of the right-hand side of the last inequality, we get $\rho_\varphi(\lambda T_n') \leqslant C_3$, $C_3 \geqslant 1$. By s-convexity of φ, $\rho_\varphi(\lambda C_3^{-1/s} T_n') \leqslant 1$. Hence $\|T_n'\|_\varphi^s \leqslant \lambda^{-s} C_3 = Cn^s \|T_n\|_\varphi^s$, where $C = 2^{(2r^2+r-2)s} C_2 C_3$.

Let us remark that taking $\varphi(u) = |u|^p$, $0 < p < 1$, we obtain the Bernstein-type inequality of Oswald [5, 2.2, p. 70].

Theorems 2 and 3 may be applied to estimate the averaged moduli of smoothness in $L_{2\pi}^\varphi$ by means of best one-sided approximations by trigonometric polynomials in $L_{2\pi}^\varphi$.

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